TIME, GAUGE, AND THE SUPERPOSITION PRINCIPLE IN QUANTUM GRAVITY

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The quantization of time-reparametrization invariant systems such as general relativity is plagued by an ambiguity relating to the role of time in the theory. If one parametrizes observables by the (unobservable) time, and then relies on the existence of an approximate "clock" degree of freedom to give physical meaning to the observables, one finds multiple quantum states that yield the same predictions yet interfere with each other.

General relativity admits a kind of Hamiltonian formulation, in which the Hamiltonian generates change in the geometry of a spacelike hypersurface with respect to an arbitrary "time" parameter τ. The arbitrariness of the time-evolution is reflected in the fact that the Hamiltonian is the sum of the supermomentum and super-Hamiltonian *constraints*. This presents no difficulty in principle for the classical theory, but gives rise to various "problems of time" in the quantum theory. (See Isham [1] and Kuchař [2].)

In this talk, we will examine the quantization of the parametrized non-relativistic particle. This system is much simpler than general relativity, and indeed it is frequently invoked as a simple example of the *efficacy* of the Dirac constraint-quantization method for time-reparametrization invariant systems. However, we will find that not only does the Dirac method fail to return us in any straightforward way to ordinary Schrödinger quantum mechanics, but it gives rise to a theory in which there are distinct quantum states with apparently identical physical content, the superposition of which yields a state with different physical content.

1 Classical parametrized particle

Consider a non-relativistic particle moving in three dimensions. If we treat t as a dynamical variable, and parametrize the motion by an arbitrary parameter $\tau = \tau(t)$, we find that we have a constraint

$$\mathcal{H} = N(p_t + H) = 0 \ , \tag{1}$$

where $N = dt/d\tau$, p_t is the "momentum" canonically conjugate to t, and $H = H(x_i, p_j)$ is the usual Hamiltonian. [a] This constraint is called the "super-Hamiltonian" and it generates change with respect to the parameter τ via the equations of motion

$$\frac{dx_i}{d\tau} = \{x_i, \mathcal{H}\} = N\frac{\partial H}{\partial p_i} \ , \ \frac{dp_i}{d\tau} = -\{p_i, \mathcal{H}\} = -N\frac{\partial H}{\partial x_i}$$

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[a] See Unruh & Wald [3] for a more thorough development of the parametrized formalism.

and

$$\frac{dt}{d\tau} = \{t, \mathcal{H}\} = N \ , \ \frac{dp_t}{d\tau} = -\{p_t, \mathcal{H}\} = 0 \ . \tag{2}$$

2 Constraint quantization

The "constraint quantization" method [4] involves turning the classical constraints into operators, and imposing them as constraints on the allowed state-vectors. Thus the first step in the constraint quantization of this system is to solve the equation $\widehat{\mathcal{H}}\Psi(x^i, t) = 0$. If we define $\hat{t} := t$ and $\hat{p}_t := -i\hbar\frac{\partial}{\partial t}$, this equation takes the form

$$i\hbar\frac{\partial}{\partial t}\Psi(x^i, t) = \widehat{H}\Psi(x^i, t) \ , \tag{3}$$

which looks rather like the Schrödinger equation. It is not—the wave functions are functions of both x^i and t. Note, too, that the equation *only* takes the functional form of the Schrödinger equation if one represents $\hat{t}$ by the multiplication operator. These issues aside, solving the equation is straightforward—one finds that solutions are of the form $\Psi(x^i, t) = e^{-(i/\hbar)\ \widehat{H}t} \circ \psi(x^i)$.

The next step is to turn these solutions into a Hilbert space. A useful way of determining the inner-product is the algebraic method of Ashtekar and Tate [5]; one selects a complete algebra of "observables" on the solution space, and one requires that the inner-product be such that these observables are self-adjoint. One choice is

$$\widehat{X}^i(0) \circ \Psi := \widehat{U}(0)\hat{x}^i\widehat{U}^{-1}(0) \circ \Psi = e^{-(i/\hbar)\widehat{H}t}\hat{x}^i \circ \psi \equiv e^{-(i/\hbar)\widehat{H}t} \circ x^i\psi(x^i)$$

and

$$\widehat{P}_i(0) \circ \Psi := \widehat{U}(0)\hat{p}_i\widehat{U}^{-1}(0) \circ \Psi = e^{-(i/\hbar)\widehat{H}t}\hat{p}_i \circ \psi \equiv e^{-(i/\hbar)\widehat{H}t} \circ -i\hbar\frac{\partial}{\partial x^i}\psi(x^i) \ , \tag{4}$$

where $\widehat{U}(0) := e^{-(i/\hbar)\widehat{H}t}$. These observables intuitively correspond to the position and momentum at some time $t = 0$. Requiring them to be self-adjoint gives an inner-product of

$$\left\langle \Psi(x^i, t), \Phi(x^i, t) \right\rangle = \int \Psi^*(x^i, t), \Phi(x^i, t) \ dx \ . \tag{5}$$

3 Observables

We see that one can straightforwardly construct one-parameter families of observables $\widehat{X}^i(t)$ and $\widehat{P}_i(t)$ that correspond, intuitively, to position and momentum at different times. This allows one to talk about the value of an observable at any given time t, as in ordinary quantum theory. However, in ordinary quantum theory, the time t is an external parameter corresponding to the classical time in which the system is embedded. If we are to consider the parametrized particle as an analogue for general relativity, then we cannot think about time in this way—there is no "external environment" in which the system is embedded. What, then, is

2

the physical content of the theory? Given a state $\Psi(x^i, t)$, one can determine the expectation value for X^1 at some time t, but this is useless if one does not have a way of physically ascertaining the time, which is, after all, not an observable.

Even in ordinary quantum theory, one doesn't measure time directly. One uses a clock, and time is determined, e.g., by the position of the hands of the clock. In this spirit, let us assume there are good "clock" variables available that give one physical, observable access to the time, so that if one knows the state $\Psi(x^i, t)$, and one knows the time, one can find expectation values for the remaining observables.

Suppose that $\widehat{X}^3(t)$ is such a "clock" variable.[b] If it is a good clock variable, it will establish fairly reliable correlations with the Schrödinger time t, and allow us to give some operational meaning to observables $\widehat{X}^i(t)$ and $\widehat{P}_i(t)$ (where i now runs from 1 to 2) parametrized by t. But now consider a wave-function $\Phi(x^i, t) = e^{-5(i/\hbar)\widehat{H}}\Psi(x^i, t)$, corresponding to a simple displacement of Ψ by five units of time. This wave-function yields exactly the *same correlations* between the clock variable $\widehat{X}^3(t)$ and all of the other observables. One would therefore like to say that it represents the *same* physical *system*, just as one would be inclined to say (along with Leibniz) that translating the entire classical world five feet to the left in Newtonian absolute space yields the same universe. The choice between Φ and Ψ looks like a choice of "gauge", which is not so surprising given that the Hamiltonian takes the form of a constraint. However, the two states interfere, and so their superposition $\alpha\Phi + \beta\Psi$ does *not* yield the same correlations. Thus we have a situation in which two physically equivalent states may be superimposed to yield a different state.

4   Conclusion

The superposition principle appears to fail for the constraint-quantized parametrized particle, and it would appear that constraint-quantized general relativity is subject to the same problem. Whereas in ordinary quantum theory, the superposition of physically equivalent states (i.e., states differing by a phase) yields a physically equivalent state, this is not the case for parametrized systems. The breakdown stems from the lack of a fiducial external observer or reference system. Lacking an external reference to give independent physical meaning to t, one must fall back on internal correlations, and this leads to the failure of the superposition principle.

References

[1] C.J. Isham, in *Integrable Systems, Quantum Groups, and Quantum Field Theories*, eds. G. Kunstatter, *et al.* (Kluwer, London, 1993).

[2] K. Kuchař, in *Proc. of the 4th Canadian Conference on General Relativity*, eds. I. Ibort and M.A. Rodriguez (World Scientific, Singapore, 1992).

[3] W. Unruh and R.M. Wald, *Phys. Rev.* D **40**, 2598 (1989).

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[b]I.e., X^3 is classically a monotonic function of t, and its "quantum fluctuations" are small enough that it is highly improbable that it will be observed to "run backward." Although such "good enough" clocks arguably suffice for coarse-grained predictions, the known absence of any quantum analogue of a perfect clock [3] may render this scheme useless at the Planck scale, which is of course the very scale at which quantum-gravitational effects are expected to come into play.

[4] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Academic, New York (1965).

[5] A. Ashtekar and R. Tate, *J. Math. Phys.* **35**, 6434 (1994).